A Formula of Landau and Mean Values of $\zeta(s)$

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Let $\rho = \beta + i\gamma$ denote a complex zero of the Riemann zeta function, $\zeta(s)$. A remarkable formula of Landau [2] (also see Titchmarsh [4;pp. 61-62]) states that for fixed $x > 1$ and $T \to \infty$

$$(1) \qquad \sum_{0<\gamma\leq T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(\log T) \quad ,$$

where $\Lambda(x) = \log p$ if $x = p^k$ for some prime p and positive integer k, and $\Lambda(x) = 0$ for all other real x. This can be proved by estimating the integral

$$(2) \qquad \frac{1}{2\pi i}\int_{\mathcal{R}} \frac{\zeta'}{\zeta}(s)\, x^s\, ds \quad ,$$

where $\mathcal{R}$ is a suitably chosen rectangle enclosing those zeros ρ for which $0 < \gamma \leq T$.

Striking as (1) is, it has little utility because it is not uniform in x. It is possible, however, by keeping the estimates of (2) explicit in x and T, to prove the following uniform version of (1).

THEOREM 1. Let $x, T > 1$. Then

$$(3) \qquad \sum_{0<\gamma\leq T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(x \log 2x \log\log 3x)$$

$$+ O(x \log 2T) + O\left(\log x \min\left(T, \frac{x}{\langle x\rangle}\right)\right)$$

$$+ O\left(\min\left(\frac{\log T}{\log x}, T \log T\right)\right) \;;$$

here $\langle x\rangle$ _is the distance from_ x _to the nearest prime power other than_ x .

Note that a trivial estimate for our sum is $\ll x\,T\log T$ ($\sqrt{x}\,T\log T$ on the Riemann hypothesis) since there are $\sim \frac{T}{2\pi}\log T$ zeros with $0<\gamma\leqslant T$. The large number of error terms in (3) reflects the varied behavior of the sum in different x ranges. Thus, the last error term is significant when x is near 1, the next-to-last when x is near a prime power. Finally, we observe that (3) has a particularly simple form if x is an integer such that $2\leqslant x\ll T$, namely

$$\sum_{0<\gamma\leqslant T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(x\log 2T\ \mathrm{loglog}\ 3T)\ .$$

Theorem 1 may be used to estimate various sums involving zeros of the zeta function. For example, one may use it to prove

THEOREM 2. _Assume the Riemann hypothesis. Let_ T _be large,_ $L=\frac{1}{2\pi}\log T$, _and_ $|\alpha|\leqslant L$ _with_ α _real. Then_

$$(4)\qquad \sum_{0<\gamma\leqslant T} |\zeta(1/2+i(\gamma+\alpha/L))|^2$$

$$= \left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T + O(T\log^{7/4} T)\ .$$

The constant implied by O _is absolute._

In [1] we gave asymptotic formulae for the sums

$$\sum_{0<\gamma\leqslant T} \zeta^{(\mu)}(\rho+i\alpha/L)\zeta^{(\nu)}(1-\rho-i\alpha/L)\qquad (\mu,\nu=0,1,\ldots)\ .$$

where $\zeta^{(\mu)}(s)$ is the μ^{th} derivative of $\zeta(s)=\zeta^{(0)}(s)$. Theorem 2 is the most interesting special case of these formulae. In fact, from (4) J. Mueller [3] deduced that

$$\lambda = \limsup_n (\gamma_{n+1} - \gamma_n)\frac{\log \gamma_n}{2\pi} > 1.9 \quad ,$$

where $0 < \gamma_1 \leq \gamma_2 \leq \ldots$ are the ordinates of the zeros of $\zeta(s)$ in the upper half-plane. Previously it was only known that $\lambda > 1$.

We now use Theorem 1 to sketch a proof of Theorem 2 which is much shorter than that given in [1]; detailed proofs of both theorems will appear elsewhere. We use the notation $A \approx B$ below to mean that $A = B +$ error terms.

We begin with the approximate functional equation for $\zeta(s)$ (see Titchmarsh [5;p.69]) from which it follows that

$$\zeta(\tfrac{1}{2} + i(\gamma + \alpha/L)) = \sum_{n \leq x} n^{-\frac{1}{2} - i(\gamma + \alpha/L)} + O(\log^{\frac{1}{4}} \gamma) \; ,$$

where $x = x(\gamma) = \gamma/2\pi\sqrt{\log \gamma}$.

Our problem is essentially to show that

$$\sum_{0 < \gamma \leq T} \left| \sum_{n \leq x} n^{-\frac{1}{2} - i(\gamma + \alpha/L)} \right|^2 \approx \left(1 - \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right) \frac{T}{2\pi} \log^2 T \; .$$

In order to avoid minor difficulties, however, we will show instead that

$$B = \sum_{0 < \gamma \leq T} \left| \sum_{n \leq X} n^{-\frac{1}{2} - (\gamma + \alpha/L)} \right|^2 \approx \left(1 - \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right) \frac{T}{2\pi} \log^2 T \; ,$$

where now $X = T/2\pi\sqrt{\log T}$ is dependent of γ.
Squaring out and changing the order of summation, we have

$$B = \sum_{m,n \leq X} \frac{1}{\sqrt{mn}} \sum_{0 < \gamma \leq T} \left(\frac{n}{m}\right)^{i(\gamma + \alpha/L)} \; .$$

The terms in (m,n) and (n,m) are conjugate, so

$$B = \sum_{n \leq X} \frac{1}{n} \sum_{0<\gamma\leq T} 1 + 2\mathcal{R}e \sum_{m<n\leq X} \frac{1}{n}\left(\frac{n}{m}\right)^{i\alpha/L} \sum_{0<\gamma\leq T} \left(\frac{n}{m}\right)^{\frac{1}{2}+i\gamma}$$

(5)

$$= B_1 + 2\,\mathcal{R}e\, B_2 ,$$

say. Since

$$\sum_{n\leq X} \frac{1}{n} \sim \log X \sim \log T ,$$

and the number of $\gamma \in (0,T]$ is $\sim \frac{T}{2\pi} \log T$,

(6) $$B_1 \sim \frac{T}{2\pi} \log^2 T .$$

Now by the Riemann hypothesis and Theorem 1, the innermost sum in B_2 equals $-\frac{T}{2\pi}\Lambda(\frac{n}{m})$ plus error terms. Hence

$$B_2 \approx -\frac{T}{2\pi} \sum_{m<n\leq X} \frac{\Lambda(\frac{n}{m})}{n} \left(\frac{n}{m}\right)^{i\alpha/L} .$$

The term in (m,n) vanishes if $m \nmid n$, so we may write

$$B_2 \approx -\frac{T}{2\pi} \sum_{km\leq X} \frac{\Lambda(k)}{k^{1-i\alpha/L}m}$$

$$\approx -\frac{T}{2\pi} \sum_{k\leq X} \frac{\Lambda(k)}{k^{1-i\alpha/L}} \sum_{m\leq X/k} \frac{1}{m}$$

$$\approx -\frac{T}{2\pi} \sum_{k\leq X} \frac{\Lambda(k)}{k^{1-i\alpha/L}} \log X/k .$$

The last sum equals

$$\int_1^X \frac{\log X/u}{u^{1-i\alpha/L}}\, d\,\Psi(u) ,$$

where $\Psi(u) = \sum_{n<u} \Lambda(n)$, and by the prime number theorem with remainder term, this is essentially

$$\int_1^X \frac{\log X/u}{u^{1-i\alpha/L}}\,du = \frac{1 + i\alpha/L \log X - X^{i\alpha/L}}{(\alpha/L)^2} .$$

Therefore,

$$2\,\mathcal{R}e\,B_2 \approx -\frac{T}{\pi}\,\frac{1-\cos(\alpha/L \log X)}{(\alpha/L)^2}$$

$$\approx -\frac{T}{2\pi}\left(\frac{\sin(\alpha/2L \log X)}{\alpha/2L}\right)^2 ,$$

or, since $\log X \sim \log T$ and $L = \frac{1}{2\pi}\log T$,

$$2\,\mathcal{R}e\,B_2 \approx -\frac{T}{2\pi}\log^2 T\left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2$$

Combining this with (5) and (6), we obtain

$$B \approx \left(1 - \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T ,$$

as desired.